

First Order Actions for New Massive Dual Gravities

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Abstract

We present a first order formulation for the fourth order action of the new massive dual gravity in four dimensions. This proposal is easily generalized to arbitrary dimension. Also, we obtain the dual actions for massless and massive Curtright fields in D dimensions.

1 INTRODUCTION

Higher order derivative theories usually contain ghost excitations, which make them non - unitary [1], although other quantum properties can be improved [2]. This is the case for gravities with curvature squared terms [3]. There are exceptions. For instance, some scalar fields coupled to gravitation admit a higher order action description ($R + R^2$ theory, see [4]). Also, the action for “Galileons” [5] contains higher derivatives, however their field equations are non-linear second order differential equations. This is analogous to the Lovelock gravity [6], whose second order field equations arise from an action which is higher order in curvature. In three dimensions, it is possible to have higher order gravity theories without loss of unitarity. The non dynamical Einstein-Hilbert action can be augmented with a third order derivative Lorentz-Chern-Simons term, the Topologically Massive Gravity theory[7], which describes the local propagation of one, ghost free and parity sensitive excitation: a massive graviton. Furthermore, the dubbed New Massive Gravity in three dimensions was formulated by Bergshoeff, Hohm and Townsend [9] four years ago. It consists, again, in the non-dynamical Einstein-Hilbert action supplemented with a specific curvature squared term, which leads to fourth order field equation. At linearized level, the New Massive Gravity is equivalent to the standard massive spin 2 Fierz-Pauli theory, which propagates two degrees of freedom with the same mass and opposite helicities ± 2 . In both cases, the Einstein-Hilbert action has a wrong sign, which is essential in order to have free ghost excitations. The New Massive Gravity has a discontinuity in its degrees of freedom when the massless limit is considered. This limit leads to the fourth order Schouten gravity, which describes a massless, conformal invariant, ghost free local excitation [10]. Also, the General Massive Gravity [11], the most general gravity theory in three dimensions, propagates two massive gravitons of helicities ± 2 but with different masses. Its gauge invariant linearized action was shown to be dual equivalent

[12] to the massive spin 2 Fierz-Pauli with a term which breaks explicitly the local Lorentz symmetry [13].

Remarkably enough, it is the observation made in [14] and [15] that solving on shell algebraic and differential constraints, it is possible to obtain in three dimensions higher order field equations for higher spin. This procedure was extended to four dimensions, but considering the massive Curtright field [16], which is the dual field to the massive spin 2 Fierz-Pauli theory in four dimensions [17], [18]. In consequence, a fourth order derivative action for the massive Curtright field is obtained. A necessary condition is that the field, which solves the constraints, belongs to the same Lorentz representation as the original field. This theory was called New Massive Dual Gravity [19]. This is an unitary theory by construction. The extension to arbitrary dimension has been achieved recently [20], [21].

The aim of this work is to present a first order formulation for New Massive Dual Gravities. The plan of this work is the following: in the next section, we introduce a first order formulation for a massless, mixed symmetry $\Phi_{mn,p}$ field, valid for arbitrary dimensions. From this action, we obtain the corresponding dual theory. In section 3, we perform the dimensional reduction of the action presented in section 2, keeping only first massive modes, to obtain the first order formulation for the massive $\Phi_{mn,p}$ field. In section 4, we propose the first order action for the New Massive Dual Gravity in four dimensions. The generalization to any dimension D will be straightforward. Throughout this work we use η_{mn} mostly positive and brackets indicate anti-symmetrization without any normalization factor, e. g. $V_{[mnp]} = V_{mnp} + V_{npm} + V_{pmn}$.

2 THE CURTRIGHT FIELD

For general mixed symmetry fields $\Phi_{M_1 M_2 \dots M_i N}$, with second order actions, Zinoviev has achieved first order actions [22]. For our purpose, we will start with the following first order action for the massless field $\Phi^{MN,P} = -\Phi^{NM,P}$ in D dimensions

$$I = \int d^D x \left(\frac{3}{4} Y_{MNP,Q} Y^{MNQ,P} - \frac{3}{4} \frac{1}{(D-3)} Y^{MN} Y_{MN} - \frac{1}{2} Y^{MNP,Q} F_{MNP,Q} \right), \quad (1)$$

where $Y^{MNP,Q}$ is an auxiliary field ($Y^{MN} = Y^{MNP,P}$) and $F_{MNP,Q}$ is the field strength of the field $\Phi^{MN,P}$

$$F_{MNP,Q} = \partial_M \Phi_{NP,Q} + \partial_N \Phi_{PM,Q} + \partial_P \Phi_{MN,Q}. \quad (2)$$

This action differs from the Zinoviev action by a change of variables:

$$Y_{MNP,Q} = \Omega_{Q,MNP} - \eta_{[QM} \Omega_{NP]} \quad (3)$$

We have invariance under the following gauge transformations

$$\delta \Phi_{MN,P} = \partial_M z_{NP} - \partial_N z_{MP}, \quad (4)$$

being the gauge parameters z_{MN} an arbitrary general second order tensor. Besides, we have invariance under local "Lorentz" transformations

$$\delta\Phi_{MN,P} = \Lambda_{MNP} \quad (5)$$

and

$$\delta Y_{MNP,Q} = \frac{1}{3} \partial_Q \Lambda_{[MNP]} - \partial_R \eta_{Q[M} \Lambda_{NP]R} \quad (6)$$

where the parameters Λ_{MNP} are completely antisymmetric.

We decompose the $\Phi^{MN,P}$ field as

$$\Phi^{MN,P} = T^{MN,P} + C^{MN,P}, \quad (7)$$

where $T^{MN,P} = -T^{NM,P}$ is the Curtright field, which satisfies the cyclic identity $T_{[MN,P]} \equiv 0$ and C_{MNP} is completely antisymmetric. The Lorentz symmetry indicates that the C_{MNP} is a non-dynamical field and the action will be written down only in terms of the Curtright field. Let us see this. The equation of motion obtained after independent variations of the auxiliary field $Y^{MNP,Q}$ is

$$Y_{Q[MN,P]} - \frac{1}{D-3} \eta_{Q[M} Y_{NP]} = F_{MNP,Q}. \quad (8)$$

This equation can be solved for $Y_{MNP,Q}$ as

$$Y_{MNP,Q} = \frac{2}{3} F_{MNP,Q} + \frac{1}{3} F_{Q[MN,P]} - \eta_{Q[M} F_{NP]}. \quad (9)$$

Substituting back into (1) we obtain the second order action

$$I = \int d^D X \left[-\frac{1}{6} F_{MNP,Q} F^{MNP,Q} - \frac{1}{4} F_{MNP,Q} F^{MNQ,P} + \frac{3}{4} F^{MN} F_{MN} \right], \quad (10)$$

where the field strength is now expressed only in terms of the Curtright field $T_{MN,P}$. Equivalently, after using the property of cyclic identity satisfied by $T_{MN,P}$ and omitting divergences, this last action can be rewritten as originally proposed by Curtright:

$$I = \frac{3}{2} \int d^D X \left[-\frac{1}{6} F_{MNP,Q} F^{MNP,Q} + \frac{1}{2} F^{MN} F_{MN} \right], \quad (11)$$

On the other hand, we can consider the field equation obtained making independent variations on $\Phi_{MN,P}$

$$\partial_Q Y^{QMN,P} = 0. \quad (12)$$

Locally this equation can be solved in terms of a $\Phi_{S_1 \dots S_{D-4}, Q}$ field as

$$Y_{PMN,Q} = \frac{1}{2} \epsilon^{PMNRS_1 \dots S_{D-4}} \partial_R \Phi_{S_1 \dots S_{D-4}, Q} \equiv \frac{1}{2(D-3)} \epsilon^{PMNS_1 \dots S_{D-3}} F_{S_1 \dots S_{D-3}, Q}. \quad (13)$$

We have defined $F_{S_1 \dots S_{D-3}, Q} \equiv \partial_{S_1} \Phi_{S_2 \dots S_{D-3}, Q} + \text{cyclic permutation}$. In five dimensions, we have $Y_{PMN, Q} \sim \epsilon^{PMNRS} \partial_R \Phi_{S, Q}$ and plugging into (1), the linearized Einstein action is obtained, illustrating the well known duality relation between Curtright and the massless spin 2 fields in five dimensions [23]. In six dimensions the Curtright field is self dual and in general D dimensions, we have the following duality relationship

$$T_{MN, P} \Leftrightarrow \Phi_{M_1 \dots M_{D-4}, N}. \quad (14)$$

The dual action is

$$\begin{aligned} I = \int d^D x [& - \frac{(D-4)}{2(D-3)} F^{M_1 \dots M_{D-3}, N} F_{M_1 \dots M_{D-3}, N} - \frac{1}{2} F_{M_1 \dots M_{D-4}, N, P} F^{M_1 \dots M_{D-4}, P, N} \\ & + \frac{1}{2(D-3)} F_{M_1 \dots M_{D-4}, N, N} F_{M_1 \dots M_{D-4}, P, P}. \end{aligned} \quad (15)$$

3 DIMENSIONAL REDUCTION

In this section we perform a dimensional reduction to the action (1) from D to D-1 dimensions in order to provide mass to the Curtright field. We will keep only the first massive modes in a similar way as was accomplished for spin 2 in [24]. For this goal, we make the following definitions:

$$Y_{mnp, q(x, y)} = \sqrt{\frac{\mu}{\pi}} Y_{mnp, q(x)} \cos \mu y, \quad (16)$$

$$Y_{mny, q(x, y)} = \sqrt{\frac{\mu}{\pi}} Y_{mn, p(x)} \sin \mu y, \quad (17)$$

$$Y_{mnp, y(x, y)} = \sqrt{\frac{\mu}{\pi}} X_{mnp(x)} \sin \mu y, \quad (18)$$

$$Y_{mny, y(x, y)} = \sqrt{\frac{\mu}{\pi}} Z_{mn(x)} \cos \mu y \quad (19)$$

and

$$\Phi_{mn, p(x, y)} = \sqrt{\frac{\mu}{\pi}} \Phi_{mn, p(x)} \cos \mu y, \quad (20)$$

$$\Phi_{my, n(x, y)} = \sqrt{\frac{\mu}{\pi}} \Phi_{mn(x)} \sin \mu y, \quad (21)$$

$$\Phi_{mn, y(x, y)} = \sqrt{\frac{\mu}{\pi}} B_{mn(x)} \sin \mu y, \quad (22)$$

$$\Phi_{my, y(x, y)} = \sqrt{\frac{\mu}{\pi}} a_{m(x)} \cos \mu y. \quad (23)$$

The dependence with the extra compact dimension is denoted by y , and low letters: m, n, p, \dots indicate coordinates in the reduced spacetime. The fields

X_{mnp} , Z_{mn} , Φ_{mn} and B_{mn} are completely antisymmetric, a_m is a vector and $Y_{mnp,q}$ and $Y_{mn,p}$ will play the role of auxiliary fields. Substituting these definitions and performing the integration on the compact coordinate we obtain the dimensionally reduced action y . The result is

$$\begin{aligned}
I = & \int d^{D-1}x \left[\frac{3}{4} Y_{mnp,q} Y^{mnq,p} - \frac{3}{4(D-3)} Y_{mnp,p} Y^{mnq,q} - \frac{3}{2} Y^{mnp,q} \partial_m \Phi_{np,q} \right. \\
& - \frac{3}{2(D-3)} Y^{mnp,p} Z_{mn} + \frac{3}{4} \frac{D-4}{D-3} Z^{mn} Z_{mn} - \frac{3}{2} Z^{mn} (\partial_m a_n - \partial_n a_m + \mu B_{mn}) \\
& - \frac{3}{2} Y^{mn,p} (\partial_m \Phi_{np} - \partial_n \Phi_{mp} - \mu \Phi_{mn,p}) - \frac{1}{2} X^{mnp} (\partial_m B_{np} + \partial_n B_{pm} + \partial_p B_{mn}) \\
& \left. + \frac{3}{2} Y^{mn,p} Y_{mp,n} - \frac{3}{2(D-3)} Y^{mp,p} Y_{mq,q} + \frac{3}{2} X^{mnp} Y_{mp,n} \right]. \quad (24)
\end{aligned}$$

This reduced action is invariant under the following gauge transformations

$$\delta \Phi_{mn,p(x)} = \partial_m z_{np(x)} - \partial_n z_{mp(x)}, \quad (25)$$

$$\delta B_{mn} = \partial_m z_{ny} - \partial_n z_{my}, \quad (26)$$

$$\delta \Phi_{mn} = \partial_m z_{yn} + \mu z_{mn} \quad (27)$$

and

$$\delta a_m = \partial_m z_{yy} - \mu z_{my}, \quad (28)$$

where we split up the gauge parameters as: $z_{MN} : (z_{mn}, z_{my}, z_{ym}, z_{yy})$.

We can break these gauge symmetries choosing appropriately the z_{mn} and z_{my} gauge parameters to fix the gauges

$$\Phi_{mn} = 0 \quad \text{and} \quad a_m = 0. \quad (29)$$

The Φ_{mn} and a_m are Stueckelberg fields. Furthermore, the reduced action inherited "Lorentz" symmetries with parameters $\Lambda_{MNP} : (\Lambda_{mnp}, \Lambda_{mny} \equiv \Lambda_{mn})$, expressed in the following transformations:

$$\delta \Phi_{mn,p} = \Lambda_{mnp}, \quad (30)$$

$$\delta \Phi_{mn} = -\Lambda_{mn} = -\delta B_{mn} \quad (31)$$

$$\delta a_m = 0, \quad (32)$$

$$\delta Y_{mnp,q} = \partial_q \Lambda_{mnp} - \partial_r \eta_{q[m} \Lambda_{np]r} - \mu \eta_{q[m} \Lambda_{np]}, \quad (33)$$

$$\delta Y_{mn,p} = \partial_p \Lambda_{mn} + \eta_{p[m} \partial_q \Lambda_{n]q}, \quad (34)$$

$$\delta X_{mnp} = -\mu \Lambda_{mnp} \quad (35)$$

and

$$\delta Z_{mn} = -\partial_p \Lambda_{pmn}. \quad (36)$$

Similarly, we can break the Lorentz symmetry associated with the Λ_{mn} parameter by choosing

$$B_{mn} = 0 \quad (37)$$

At this stage, we can eliminate the Z_{mn} field since it appears as a quadratic multiplier. Its equation of motion leads to determining its value:

$$Z_{mn} = \frac{1}{D-4} Y_{mnp,p}. \quad (38)$$

With these gauges fixing and substituting this value of Z_{mn} , the reduced action boils down to

$$\begin{aligned} I = & \int d^D x \left[\frac{3}{4} Y_{mnp,q} Y^{mnq,p} - \frac{3}{4(D-3)} Y_{mnp,p} Y^{mnq,q} \right. \\ & - \frac{3}{2} Y^{mnp,q} \partial_m \Phi_{np,q} + \frac{3}{2} X^{mnp} Y_{np,m} \\ & \left. + \frac{3}{2} Y^{mn,p} Y_{mp,n} - \frac{3}{2(D-2)} Y^{mp,p} Y_{mq,q} + \frac{3\mu}{2} Y^{mn,p} \Phi_{mn,p} \right]. \quad (39) \end{aligned}$$

The Lorentz symmetry associated with the parameter Λ_{mnp} can be used to fix gauge in two different ways. The first alternative consists in choosing $X^{mnp} = 0$ and the other possibility allows us to eliminate the antisymmetric part of $\Phi_{mn,p}$ (i. e. $\Phi_{[mn,p]} \equiv 0$). Both choices will lead to the same result: the first order formulation for the massive Curtright action. Fixing the gauge $X^{mnp} = 0$, the field equation for the $Y^{mn,p}$ is

$$Y_{mp,n} - Y_{np,m} - \frac{1}{D-2} (\eta_{pn} Y_m - \eta_{pm} Y_n) = -\mu \Phi_{mn,p}, \quad (40)$$

which can be solved as

$$Y_{mn,p} = -\frac{\mu}{2} [\Phi_{mn,p} + \Phi_{mp,n} - \Phi_{np,m}] - \mu [\eta_{mp} \Phi_n - \eta_{np} \Phi_m] \quad (41)$$

and when this expression is plugged into (39), the result is the following non-gauge invariant action

$$\begin{aligned} I = & \int d^D x \left[\frac{3}{4} Y_{mnp,q} Y^{mnq,p} - \frac{3}{4(D-3)} Y_{mnp,p} Y^{mnq,q} - \frac{3}{2} Y^{mnp,q} \partial_m \Phi_{np,q} \right. \\ & \left. - \frac{3\mu^2}{8} (\Phi^{mn,p} \Phi_{mp,n} + 2\Phi^{mn,p} \Phi_{mp,n} - 4\Phi^m \Phi_m) \right]. \quad (42) \end{aligned}$$

The first line in this dimensional reduced action (42) is the first order action for the massless Curtright field. Now, if we decompose the $\Phi_{mn,p}$ in its irreducible parts : $\Phi_{mn,p} = T_{mn,p} + C_{mnp}$, with $T_{[mn,p]} \equiv 0$ and C_{mnp} completely antisymmetric, this part does not depends on C_{mnp} , while the coefficients of the massive term, the second line in action (42), are such that the constituents of $\Phi_{mn,p}$ are decoupled. Indeed

$$\Phi^{mn,p} \Phi_{mn,p} + 2\Phi^{mn,p} \Phi_{mp,n} - 4\Phi^m \Phi_m = 2T_{mn,p} T^{mn,p} - 4T_m T^m - C_{mnp} C^{mnp}. \quad (43)$$

In consequence, C_{mnp} is a non dynamical field, and then by eliminating $Y_{mnp,q}$ through its equation of motion, we obtain the second order action for the massive Curtright field.

$$I = \frac{3}{2} \int d^D x \left[-\frac{1}{6} F_{mnp,q} F^{mnp,q} + \frac{1}{2} F^{mn} F_{mn} - \frac{1}{2} (T_{mn,p} T^{mn,p} - 2T_m T^m) \right]. \quad (44)$$

The number of degrees of freedom of this theory is given by $\frac{1}{3}(D-1)(D+1)(D-3)$, while in the massless case the Curtright field propagates $\frac{1}{3}D(D-2)(D-4)$ transverse modes. The massless Curtright field in $D+1$ dimensions has the same number of degrees of freedom as the massive Curtright field in D dimensions.

If we have chosen the gauge $C_{mnp} = 0$ ($\Phi_{mn,p} = T_{mn,p}$), then X^{mnp} is a multiplier which tells us that $Y_{[mn,p]} = 0$. After eliminating $Y_{mn,p}$ through the use of its equation of motion, we again reach the action for the massive Curtright field. From now on, we will refer only to the Curtright $T_{mn,p}$.

From the first order action (42), we can obtain the dual theory for the massive Curtright field if we eliminate the $\Phi^{mn,p}$ field through its equation of motion. In fact, we obtain

$$\Phi^{mn,p} = -\frac{1}{\mu^2} \partial_q (Y^{qpm,n} - Y^{qpn,m}) + \frac{1}{\mu^2(D-2)} \partial_q (\eta^{pm} Y^{qn} - \eta^{pn} Y^{qm}) \quad (45)$$

and substituting into (42), the following action for $Y^{mnp,q}$ is obtained

$$\begin{aligned} \frac{2}{3} \mu^2 I &= \int d^D x \left[\partial_q Y^{qmn,p} \partial^r Y_{rmp,n} - \frac{1}{D-2} \partial_q Y^{qm} \partial_r Y^{rm} \right. \\ &\quad \left. + \frac{\mu^2}{2} (Y^{mnp,q} Y_{mnq,p} - \frac{1}{D-3} Y^{mn} Y_{mn}) \right]. \end{aligned} \quad (46)$$

Now, if we decompose $Y^{mnp,q}$ in its trace and traceless parts

$$Y^{mnp,q} = W^{mnp,q} + \frac{1}{D-2} (\eta^{qm} Y^{np} + \eta^{qn} Y^{pm} + \eta^{qp} Y^{mn}), \quad (47)$$

where $W^{mnp,q}$ is the traceless part of $Y^{mnp,q}$ ($W_p^{mnp} = 0$), the action is written down as

$$\frac{2}{3} \mu^2 I = \int d^D x \left[\partial_q W^{qmn,p} \partial^r W_{rmp,n} + \frac{\mu^2}{2} (W^{mnp,q} W_{mnq,p} - \frac{1}{(D-2)(D-3)} Y^{mn} Y_{mn}) \right], \quad (48)$$

which clearly shows that the trace of $Y^{mnp,q}$ is a non dynamical variable. Next, we introduce the dual of $W^{mnp,q}$:

$$W^{mnp,q} = \frac{1}{(D-3)} \epsilon^{mnp r_1 \dots r_{D-3}} T_{r_1 \dots r_{D-3} q}. \quad (49)$$

Since $W^{mnp,q}$ is traceless, then $T_{r_1 \dots r_{D-3}, q}$ satisfy the cyclic identity: $T_{[r_1 \dots r_{D-3}, q]} \equiv 0$ and introducing the field strength $F_{r_1 r_2 \dots r_{D-2}, q} \equiv \partial_{r_1} T_{r_2 \dots r_{D-3}, q} + \text{cyclic per-}$

mutations, the dual action for the massive Curtright field is

$$I = \int d^D x \left[- \frac{1}{(D-2)} F^{r_1 \dots r_{D-3} p, q} F_{r_1 \dots r_{D-3} p, q} - F_{r_1 \dots r_{D-3} p, p} F^{r_1 \dots r_{D-4} q, q} \right. \\ \left. - \mu^2 (T_{r_1 \dots r_{D-3}, p} T^{r_1 \dots r_{D-3}, p} - (D-3) T_{r_1 \dots r_{D-4} p, p} T^{r_1 \dots r_{D-4} p, p}) \right] \quad (50)$$

Thus, we have established the duality relationship between the massive Curtright field ($T_{mn,p}$) and a massive mixed symmetry field $T_{m_1 \dots m_{D-3}, p}$

4 THE FIRST ORDER ACTION

In four dimensions the massless Curtright does not propagate any local excitations like the massless spin 2 in three dimensions. Generally, a massless $\Phi_{m_1, m_2, m_{D-2}, n}$ field in D dimensions does not have any degrees of freedom. In four dimensions, Zinoviev has written down a first order action for the non dynamical Curtright field, akin to the well known first order action of linearized Einstein action in three dimensions. This action is expressed in terms of the $T_{mn,p}$ field and an auxiliary field h_{mn} neither symmetric nor antisymmetric. This action is written out as

$$I_C = \int d^4 x \left[-\frac{1}{4} (h_{mn} h^{nm} - h^2) + \frac{1}{2} \epsilon^{mnpq} h_{mr} \partial_n T_{pq, r} \right]. \quad (51)$$

In fact, the field equation obtained by varying the auxiliary field h^{nm} tells us:

$$h^{nm} = \epsilon^{mpqr} \partial_n T_{pq, r}. \quad (52)$$

Note that $h = 0$ on shell, but we will keep the h^2 term in the action because it will be necessary to have the dual off-shell equivalence between massive spin 2 and Curtright fields in four dimensions. When this value of h_{nm} is substituted into I_C , the second order Curtright action is obtained

$$I_C = \frac{1}{2} \int d^4 x T_{mn, p} G^{mn, p}, \quad (53)$$

where

$$G^{mn, p} \equiv \frac{1}{2} \epsilon^{mnsr} \epsilon^{pquv} \partial_r \partial_q T_{uv, s} \quad (54)$$

is the "generalized Einstein tensor" for the Curtright field introduced in ref [19].

Now, we are ready to propose the first order action in four dimensions for the fourth order "new massive gravity". This action involves three independent variables: the Curtright field $T_{mn,p}$, a h_{mn} field and an auxiliary field $Y_{mn,p}$. Both $T_{mn,p}$ and $Y_{mn,p}$ satisfy cyclic identities. The action is written down as:

$$I = \int d^4 x \left[- \frac{1}{4} (h_{mn} h^{nm} - h^2) + \frac{1}{2} \epsilon^{mnpq} h_{mr} \partial_n T_{pq, r} \right. \\ \left. + Y^{mn, p} Y_{mp, n} - \frac{1}{2} Y^m Y_m + \mu Y^{mn, p} T_{mn, p} \right]. \quad (55)$$

Making independent variations on the fields, we obtain

$$\frac{\delta I}{\delta h_{mn}} = 0 \quad \Rightarrow \quad h^{mn} = \epsilon^{npqr} \partial_p T_{qr,m}, \quad (56)$$

$$\frac{\delta I}{\delta Y_{mn,p}} = 0 \quad \Rightarrow \quad Y_{mn,p} = -\frac{\mu}{2} [T_{mn,p} + T_{mp,n} - T_{np,m}] - \mu [\eta_{mp} T_n - \eta_{np} T_m] \quad (57)$$

and

$$\frac{\delta I}{\delta T_{mn,p}} = 0 \quad \Rightarrow \quad Y^{mn,p} = -\frac{1}{2\mu} \epsilon^{mnqr} \partial_q h_{rp}. \quad (58)$$

The equations (56) and (58) are first order in derivatives. As usual, the auxiliary fields (h_{mn} and $Y_{mn,p}$) can be substituted, using equations (56) and (57), in order to reach the second order action (44) for the massive Curtright field $T_{mn,p}$.

Alternatively, we can express the $Y_{mn,p}$ field (eq. (58)) in terms of the second derivative of the Curtright field $T_{mn,p}$ using eq. (56):

$$Y_{mn,p} = -\frac{1}{\mu} G_{mn,p[T]}. \quad (59)$$

Since $Y_{mn,p}$ satisfies a cyclic identity, we can write

$$Y^{mn,p} Y_{mp,n} - \frac{1}{2} Y^m Y_m = \frac{1}{2} Y^{mn,p} Y_{mn,p} - \frac{1}{2} Y^m Y_m. \quad (60)$$

Plugging into the first order action (55), the values of h_{mn} and $Y_{mn,p}$ given by equations (56) and (59), we obtain the fourth order action of the new massive gravity in four dimensions :

$$I = \frac{1}{2} \int d^4 x [-T_{mn,p} G^{mn,p} + \frac{1}{\mu^2} G^{mn,p} S_{mn,p}], \quad (61)$$

where

$$S_{mn,p} \equiv G_{mn,p} - \frac{1}{2} (\eta_{np} G_m - \eta_{mp} G_n) \quad (62)$$

is a generalized "Schouten" tensor. Note the the wrong sign of the kinematical Curtright term, it has been flipped over. In this way we have implemented a first order parent action which establishes the equivalence between the second order massive Curtright theory and the higher order action of [19].

Moreover, we can establish the dual equivalence between massive Curtright field and the massive spin 2 field, in four dimensions, from our parent first order action. Indeed, using only (58), we obtain:

$$I = \int d^4 x [-\frac{1}{16} (C_{mn,p} C^{mn,p} - \frac{1}{16} C_{mn,p} C^{mp,n} + \frac{1}{4} C_m C^m) - \frac{1}{4} (h_{mn} h^{nm} - h^2)], \quad (63)$$

where $C_{mn,p} \equiv \partial_m h_{np} - \partial_n h_{mp}$ and $C_m = C_{mn,n}$. This is just the massive second order spin 2 Fierz-Pauli action.

The generalization to D dimensions is

$$\begin{aligned}
I = & \int d^D x \left[-\frac{1}{4} (h_{mn} h^{nm} - h^2) + \frac{1}{2} \epsilon^{mn s_1 \dots s_{D-2}} \omega_{mr} \partial_n T_{s_1 \dots s_{D-2}, r} \right. \\
& + Y^{m_1 \dots m_{D-3} n, p} Y_{m_1 \dots m_{D-3} n, p} - \frac{1}{D-2} Y^{m_1 \dots m_{D-3}} Y_{m_1 \dots m_{D-3}} \\
& \left. + \mu Y^{m_1 \dots m_{D-2}, p} T_{m_1 \dots m_{D-2}, p} \right], \tag{64}
\end{aligned}$$

where $T_{m_1, m_2, \dots, m_{D-2}, n}$ is the dual massive field of the massive h_{mn} field [18]. $Y_{m_1, m_2, \dots, m_{D-2}, n}$ is an auxiliary field ($Y_{m_1, m_2, \dots, m_{D-3}} = Y_{m_1, m_2, \dots, m_{D-3} n, n}$). Both $T_{m_1, m_2, m_{D-2}, n}$ and $Y_{m_1, m_2, m_{D-2}, n}$ satisfy cyclic identities. Now, we have the following equations

$$h^{mn} = \epsilon^{n p r_1 \dots r_{D-2}} \partial_p T_{r_1 \dots r_{D-2}, m} \tag{65}$$

and

$$Y^{m_1 \dots m_{D-2}, n} = -\frac{1}{\mu} G^{m_1 \dots m_{D-2}, n}, \tag{66}$$

where

$$G^{m_1 \dots m_{D-2}, n} = \frac{1}{2} \epsilon^{m_1 \dots m_{D-2} q p} \epsilon^{n s r_1 \dots r_{D-2}} \partial_q \partial_s T_{r_1 \dots r_{D-2}, p}, \tag{67}$$

is the generalized Einstein tensor. Plugging (65) and (66) into (64), the fourth order action ([20], [21]) is obtained

$$I = \int d^D x \left[\frac{1}{2} T_{m_1 \dots m_{D-2}, n} G^{m_1 \dots m_{D-2}, n} + \frac{1}{\mu^2} G^{m_1 \dots m_{D-2}, n} S_{m_1 \dots m_{D-2}, n} \right], \tag{68}$$

where

$$S_{m_1 \dots m_{D-2}, n} = G_{m_1 \dots m_{D-2}, n} - \frac{1}{D-2} \eta_{n[m_{D-2}} G_{m_1 \dots m_{D-3}] p, p}, \tag{69}$$

is the generalized Schouten tensor. Also, the first order action (64) permits to show the dual equivalence

$$h_{mn} \Leftrightarrow T_{m_1 \dots m_{D-2}, n}, \tag{70}$$

which was established in ([18]) for the massive linearized gravitation in arbitrary dimensions.

Finally, we consider the case for three dimensions. We must expect to recover the fourth order new massive gravity [9]. The first order action to consider is

$$\begin{aligned}
I = \int d^3 x \left[-\frac{1}{2} (\omega_{mn} \omega^{nm} - \omega^2) + \epsilon^{mnpq} \omega_{mr} \partial_n e_{pq} \right. \\
\left. + \frac{1}{2} Y^{mn} Y_{nm} - \frac{1}{4} Y^2 + \mu Y^{mn} e_{mn} \right]. \tag{71}
\end{aligned}$$

We have denoted the auxiliary fields by ω_{mn} (for convenience) and Y_{nm} , which are general second order tensor. The coefficients have been slightly changed in

concordance with the result of ref [24]. The following equations of motion are obtained

$$\frac{\delta I}{\delta \omega_{nm}} = 0 \quad \Rightarrow \quad \omega_{mn} - \eta_{nm} + \epsilon_n^{pq} \partial_p e_{qn} = 0, \quad (72)$$

$$\frac{\delta I}{\delta Y_{mn}} = 0 \quad \Rightarrow \quad Y_{mn} - \frac{1}{2} \eta_{mn} Y + \mu e = 0 \quad (73)$$

and

$$\frac{\delta I}{\delta e_{mn}} = 0 \quad \Rightarrow \quad Y_{mn,p} = -\frac{1}{\mu} \epsilon_m^{pq} \partial_p \omega_{qn}. \quad (74)$$

The equation (72) is solved as:

$$\begin{aligned} \omega_{mn} &= \epsilon_n^{pq} \partial_p e_{qm} - \frac{1}{2} \eta_{mn} \epsilon^{pqr} \partial_p e_{qr} \\ &\equiv W_{mn[e]} \end{aligned} \quad (75)$$

while the equation (73) determines Y_{mn}

$$Y_{mn} = -\mu(e_{mn} - \eta_{mn} e). \quad (76)$$

Inserting equations (75) and (76) into the equation (71), the following action is obtained

$$I = \int d^3x \left[\frac{1}{2} e_{mq} \epsilon^{mnp} \partial_n W_{pq[e]} - \frac{\mu^2}{2} (e_{mn} e^{nm} - e^2) \right], \quad (77)$$

and after introducing the symmetric and antisymmetric parts of e_{mn} ($e_{mn} = h_{mn} + \epsilon_{mnp} v^q$), we recognize (77) as the action for the massive spin 2. The first term is the linearized Einstein action ($e_{mq} \epsilon^{mnp} \partial_n W_{pq[e]} = h_{mn} G^{mn}$, $G^{mn} = R_{mn} - \frac{1}{2} \eta_{mn} R$ is the linearized Einstein tensor) expressed only in terms of the symmetric part h_{mn} of e_{mn} . The antisymmetric component v^q appears decoupled in the massive term. Alternatively, as in the formulation for the Curtright field, we can express Y_{mn} as

$$Y_{mn} = -\frac{1}{\mu} \epsilon_m^{pq} \partial_p W_{qn} = -\frac{1}{\mu} G_{mn[h]} \quad (78)$$

and substituting (75) and (78) into the action, the new massive gravity is reached:

$$I_{BHT} = \frac{1}{2} \int d^3x \left[-h_{mn} G^{mn} + \frac{1}{\mu^2} (R_{mn} R^{mn} - \frac{3}{8} R^2) \right]. \quad (79)$$

Thus, we have implemented a first order formulation for the fourth order new massive gravity in three dimensions. On the other hand, the action expressed only in terms of ω_{mn} is the same form as for the e_{mn} . In three dimensions, the massive spin 2 field is self-dual.

5 CONCLUSIONS

In this paper we have found a first order action for the fourth order action of the New Massive Gravity in four dimensions. In order to achieve this goal, two auxiliary fields were introduced, besides the Curtright field. Its extension to arbitrary dimensions was straightforward formulated, including $D = 3$ new massive gravity. It was necessary to consider a first order kinematical term for a general mixed symmetry $T_{s_1 \dots s_{D-2}, r}$ [17]:

$$I = \int d^D x \left[-\frac{1}{4} (h_{mn} h^{nm} - h^2) + \frac{1}{2} \epsilon^{mns_1 \dots s_{D-2}} \omega_{mr} \partial_n T_{s_1 \dots s_{D-2}, r} \right]. \quad (80)$$

Furthermore, we have established the dual actions for massless and massive Curtright field in any dimensions:

$$T_{mn,p} \Leftrightarrow T_{m_1 \dots m_{D-4}, p} (\mu = 0) \quad T_{mn,p} \Leftrightarrow T_{m_1 \dots m_{D-3}, p} (\mu \neq 0). \quad (81)$$

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